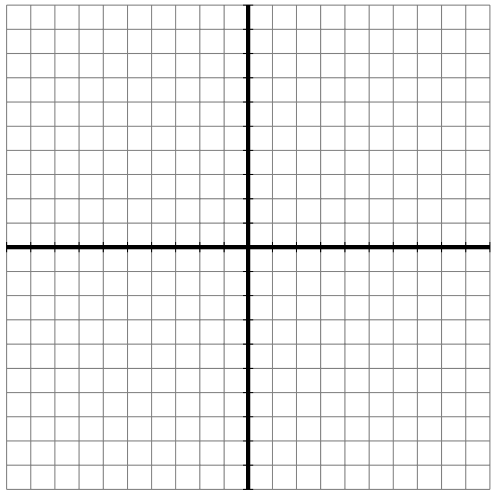
"Coordinates" probably makes you think of the points on the following "standard" coordinate grid.

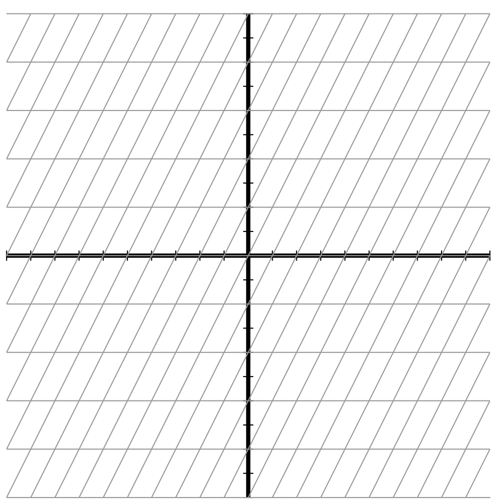


We can represent points in the plane using our standard basis for $\mathbb{R}^2$, $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and plot these points directly, or we could even plot a linear combination.

For example, let $\bar{x} = 2\bar{e}_1 + 6\bar{e}_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$. We can plot this on the above axes.

There are often applications where the coordinate system above doesn't quite represent what we are studying. For example, in Calculus you study polar coordinates.

Let's describe a different coordinate system. Say $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$, where $\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\bar{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The "$\mathcal{B}$ coordinate grid" is drawn below.



Where should $\bar{x}$ be on this grid? Well, notice that $\bar{x} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\bar{b}_1 + 3\bar{b}_2$. Let's plot this on the $\mathcal{B}$-grid. Does it land where we expect?

It seems like the *coefficients* of our basis vectors corresponds to the new "coordinates", that is, $(-1, 3)$ are the "new" coordinates for $\bar{x}$ interpreted using $\mathcal{B}$. But how do we find the coordinates in general if given a different basis?

Theorem: (**The Unique Representation Theorem**) Let $\mathcal{B} = \{\bar{b}_1, \dots, \bar{b}_n\}$ be a basis for a vector space V. Then for each $\bar{x}$ in V, there exists a set of scalars $\{c_1, c_2, \dots, c_n\}$ such that

$$\bar{x} = c_1\bar{b}_1 + \cdots + c_n\bar{b}_n$$

Proof.

This theorem is incredibly important so that we can define coordinate systems for vector spaces.

Definition: Suppose $\mathcal{B} = \{\bar{b}_1, \dots, \bar{b}_n\}$ is a basis for V, and let $\bar{x}$ be in V. The (or the **coordinates of $\bar{x}$ relative to basis $\mathcal{B}$**) are the weights

$$c_1, \dots, c_n$$

so that $\bar{x} = c_1\bar{b}_1 + \cdots + c_n\bar{b}_n$.

If $c_1, \dots, c_n$ are the $\mathcal{B}$-coordinates of $\bar{x}$, then we let $[\bar{x}]_{\mathcal{B}}$ denote the vector

$$[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

which we call the (relative to $\mathcal{B}$), or the **$\mathcal{B}$-coordinate vector of $\bar{x}$**.

Example: Let $\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\bar{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$ is a basis for $\mathbb{R}^2$. Find $\bar{x}$ so that $[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Example: Let $\mathcal{B} = \{\bar{e}_1, \bar{e}_2\}$ be the standard basis for $\mathbb{R}^2$. Find $[\bar{x}]_{\mathcal{B}}$ for $\bar{x} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$.

Given the coordinate vector of $\bar{x}$, it's quick to compute the linear combination. However, it's a little more challenging to guess the coefficients for a linear combination if your basis isn't the standard basis

Example: Let $b_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$, and $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$. $\mathcal{B}$ is a basis for $\mathbb{R}^2$. Find the coordinate vector $[\bar{x}]_{\mathcal{B}}$ of $\bar{x}$ relative to $\mathcal{B}$.

This example motivates the following definition.

Definition: If $\mathcal{B} = \{\bar{b}_1, \dots, \bar{b}_n\}$ is a basis for $\mathbb{R}^n$, the
denoted $P_{\mathcal{B}}$, is given by

$$P_{\mathcal{B}} =$$

In this way, if $\bar{x} = c_1\bar{b}_1 + \cdots + c_n\bar{b}_n$, we then have

$$\bar{x} =$$

Remark: $P_{\mathcal{B}}$ must be invertible! It's columns are linearly independent, and it's a square matrix, so the invertible matrix theorem applies.

Multiplying $\bar{x} = P_{\mathcal{B}}[\bar{x}]_{\mathcal{B}}$ on both sides gives us a formula for $[\bar{x}]_{\mathcal{B}}$:

$$[\bar{x}]_{\mathcal{B}} =$$

We use this to build a useful linear transformation. Let $T(\bar{x}) = P_{\mathcal{B}}{}^{-1}\bar{x} = [\bar{x}]_{\mathcal{B}}$. By definition, T sends $\bar{x}$ to its coordinates relative to $\mathcal{B}$. We call this the **coordinate mapping (determined by $\mathcal{B}$).** Sometimes we'll suppress the notation, writing $\bar{x} \mapsto [\bar{x}]_{\mathcal{B}}$.

Example: Let $b_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$, and $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$. $\mathcal{B}$ is a basis. Find the coordinate vector $[\bar{x}]_{\mathcal{B}}$ of $\bar{x}$ relative to $\mathcal{B}$ by using and inverse matrix ${P_{\mathcal{B}}}^{-1}$.

The Coordinate Mapping

Understanding the coordinate mapping will help us see how we use coordinates in a useful way for our purposes.

Example: Let $\mathcal{B}$ be the standard basis for $\mathbb{P}_3$, that is $\mathcal{B} = \{1, t, t^2, t^3\}$. If $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -4 \\ 9 \\ 0 \end{bmatrix}$ determine $p(t)$.

Example: Let $\mathcal{B}$ be as before. Find the coordinate vector of $[p(t)]_{\mathcal{B}}$, the coordinate vector of $p(t)$ relative to $\mathcal{B}$, if

1. $p(t) = 1 + 6t + 4t^2$.

2. $p(t) = -3 + 2t^2 + t^3$

3. $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$.

Observation: The take-away is that every polynomial in $\mathbb{P}_3$ gives a vector in $\mathbb{R}^4$, and each vector in $\mathbb{R}^4$ is assigned a *unique* polynomial in $\mathbb{P}_3$. But the similarities go farther!

Example: Let $p(t) = 1 + 6t + 4t^2$ and $q(t) = -3 + 2t^2 + t^3$, and let $\mathcal{B}$ be as before. Find $[p(t) + q(t)]_{\mathcal{B}}$ and $[2 \cdot p(t)]_{\mathcal{B}}$, and compare both to $[p(t)]_{\mathcal{B}}$ and $[q(t)]_{\mathcal{B}}$.

Observation: It appears that as vector spaces, $\mathbb{P}_3$ and $\mathbb{R}^4$ are virtually indistinguishable! Adding and scaling essentially does the same thing in both situations! Basically, both spaces are the same, and just represent the same information differently.

Theorem: Let $\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be a basis for V. The coordinate mapping $\bar{x} \mapsto [\bar{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

The fact that it is one-to-one follows from the fact that every vector is determined by its coordinates by definition. (In context of the prior page, this means that each polynomial determined the vector.)

Likewise, the map must be onto—this follows from the face that ${P_{\mathcal{B}}}^{-1}\bar{x} = [\bar{x}]_{\mathcal{B}}$ has at least one solution by the Invertible Matrix Theorem. (In context of the prior page, this means every vector in $\mathbb{R}^4$ is mapped to.

Through a bit of work, one can show the *very* surprising fact that its a linear transformation, that is,

$$[r_1\bar{x}_1 + r_2\bar{x}_2 + \cdots + r_p\bar{x}_p]_{\mathcal{B}} =$$

On the prior page, this was showing that adding or scaling polynomials does the same thing to their corresponding vectors.

We remarked that $\mathbb{P}_3$ and $\mathbb{R}^4$ basically are the same space, just represented differently. This is captured in the following definition.

Definition: A linear transformation from V to W that is both onto and one-to-one is called an

Our map $\bar{x} \mapsto [\bar{x}]_{\mathcal{B}}$ is an isomorphism from a vector space V to $\mathbb{R}^n$. This tells us that even though the objects of V may *not* be vectors in $\mathbb{R}^n$, we can really think of them as vectors in $\mathbb{R}^n$.

On the prior page, our vector space was $\mathbb{P}_3$. So we've displayed an isomorphism between $\mathbb{P}_3$ and $\mathbb{R}^4$, which is the math way of saying these are the same as vector spaces.

This is useful, because we have lots of tools for $\mathbb{R}^n$ which can now be extended to vector spaces.

Example: Use coordinate vectors to verify that $1+2t^2$, $4+t+5t^2$, and $3+2t$ are linearly dependent in $\mathbb{P}^2$.

Remark: The power here is that a computer is very good at row reducing matrices, and so converting this to a problem about vectors instead of polynomials makes things easier to figure out.

Example: Use coordinate vectors to verify that 1, t, and t^2 are linearly independent in $\mathbb{P}^2$.

Example: The set $\mathcal{B} = \{1-t^2, t-t^2, 2-2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $p(t) = 3+t-6t^2$ relative to $\mathcal{B}$ using an inverse matrix $P_{\mathcal{B}}{}^{-1}$.