Learning Objective

- Determine if a set of vectors is a basis for $\mathbb{R}^n$
- Determine if a set of vectors is a basis for a subspace
- Understand how to find a basis for Nul A, Col A, or other subspaces

Linearly Independent Sets; Bases

Definition: An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

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has __________ the trivial solution, $c_1 = 0, \ldots, c_p = 0$.

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist scalars $c_1, \ldots, c_p$, *not all zero*, such that

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Some facts from Section 1.7 still hold true for general vector spaces:

- A set containing a single vector $\mathbf{v}$ is linearly ________________________ if and only if $\mathbf{v} \neq \mathbf{0}$.
- A set of two vectors is **linearly dependent** if and only if one of the vectors is a

  ____.
- Any set containing __________________________________ is **linearly dependent**.

Theorem 4.4: An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if $\mathbf{v}_j$ for some $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a ____ set, and

(ii) $H =$ ____.

Remark: The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of the $n \times n$ identity matrix, I_n, is called the ____________________ basis for $\mathbb{R}^n$.

The Spanning Set Theorem

A basis can be constructed from a spanning set of vectors by discarding vectors which are linear combinations of preceding vectors in the indexed set.

Theorem 4.5 (The Spanning Set Theorem):

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V and let $H = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

(a) If $\mathbf{v}_k$ for some $1 \leq k \leq p$ can be written as a linear combination of the other vectors $\mathbf{v}_j$ for $j \neq k$ in S, then the set formed by ______________________ from S still spans H.

(b) If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Theorem 4.6: The ____________________________ of a matrix A form a basis for Col A.

Summary

- To find a basis for Nul A, use elementary row operations to transform $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ to its reduced row echelon form $\begin{bmatrix} B & \mathbf{0} \end{bmatrix}$. Use the reduced row echelon form to find the parametric vector form of the solution set to $A\mathbf{x} = \mathbf{0}$. The vectors found in this parametric form of the solution set are a basis for Nul A.

- To find a basis for Col A, use elementary row operations to reduce A to echelon form and identify the pivot columns of A.

Warning: We must row reduce A to an echelon form in order to identify the pivot columns of A. However, be careful to use *the pivot columns of A itself* for the basis for Col A (**not** the pivot columns of an echelon form of A).