Learning Objective

- Understand how to find the null space of an $m \times n$ matrix
- Understand how to find the column space of an $m \times n$ matrix
- Determine if a vector is in Nul A or Col A
- Understand how to find a nonzero vector in Nul A or Col A
- Understand how the kernel of a linear transformation relates to the null space
- Understand how the range of a linear transformation relates to the column space

The Null Space of a Matrix

Definition: The ________________________ of an $m \times n$ matrix A, denoted Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A =$$

Remark: We can think of Nul A as the set of all $\mathbf{x}$ in $\mathbb{R}^n$ that are mapped into the zero vector of $\mathbb{R}^m$ via the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Theorem 4.2: The null space of an $m \times n$ matrix A is a __________________ of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Proof: Since A has n columns, the solutions to $A\mathbf{x} = \mathbf{0}$ are in ______. Therefore, Nul A is a subset of ______. We must check that Nul A satisfies the three properties of a subspace.

1.

2.

3.

An Explicit Description of Nul A

Solving the equation ________________ produces an explicit description of Nul A.
Step 1: Find the general solution of $A\mathbf{x} = \mathbf{0}$.
Step 2: Write the solution in parametric vector form.
Step 3: Use the vectors from Step 2 as a spanning set for Nul A.

Remarks:

- The spanning set produced by this method is linearly independent.
- When Nul A contains nonzero vectors, the number of vectors in the spanning set for Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

The Column Space of a Matrix

Definition: The __________________________ of an $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_\mathbf{n} \end{bmatrix}$, then

Col $A =$

Theorem 4.3: The column space of an $m \times n$ matrix A is a ___________________ of $\mathbb{R}^m$.

The Kernel and Range of a Linear Transformation

Definition: A _______________________________________ T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

(i) ________________________________ for all $\mathbf{u}, \mathbf{v}$ in V, and

(ii) ________________________________ for all $\mathbf{u}$ in V and all scalars c.

The **kernel** of T is the set of all _____________ such that ___________________ (the zero vector in W).

The **range** of T is the set of all __________________________ of the form __________ for some _____________.