Learning Objective

- Determine if a set is a subspace of a vector space
- Understand how to find a spanning set of vectors for a subspace of $\mathbb{R}^n$

Vector Spaces

Many concepts concerning vectors in $\mathbb{R}^n$ can be extended to other mathematical systems.

Definition: A ________________________________ is a nonempty set V of objects along with two operations, called *addition* and *scalar multiplication*, such that the following axioms hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and for all scalars c and d.

1. _____________ is in V (**closed under addition**)
2. $\mathbf{u} + \mathbf{v} =$ _____________ (**commutativity**)
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} =$ ___________________ (**associativity**)
4. There exists an element _____ in V such that ____________ $= \mathbf{u}$ (**additive identity**)
5. For every $\mathbf{u}$ in V, there is a vector ______ in V such that _____________ $= \mathbf{0}$ (**additive inverses**)
6. __________ is in V (**closed under scalar multiplication**)
7. $c(\mathbf{u} + \mathbf{v}) =$ ___________________ (**distributive property for vector addition**)
8. $(c + d)\mathbf{u} =$ ___________________ (**distributive property for scalar addition**)
9. $c(d\mathbf{u}) =$ _____________ (**associativity of scalar multiplication**)
10. $1\mathbf{u} =$ ______ (**multiplicative identity**)

The elements of V are called ___________________. For now, we will assume that our scalar field is $\mathbb{R}$, i.e. our vector spaces will be "real" vector spaces.

Remark: Using only these axioms, we can show that the zero vector is unique and additive inverses are unique.

For each $\mathbf{u}$ in V and scalar c,

$$0\mathbf{u} = \mathbf{0}$$

$$c\,\mathbf{0} = \mathbf{0}$$

$$-\mathbf{u} = (-1)\mathbf{u}$$

Subpaces

Special subsets of vector spaces are vector spaces themselves.

> **Definition:** A ____________________ of a vector space V is a subset H of V that has three properties:
>
> 1. The zero vector of V ____________________.
> 2. H is closed under ______________________________. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum _____________ is in H.
> 3. H is closed under ____. That is, for each $\mathbf{u}$ in H and each scalar c, the vector __________ is in H.

Remark: If the subset H satisfies these three properties, then H itself is a __________________________.

Subpaces Spanned by Sets

Recall: A **linear combination** refers to any sum of scalar multiples of vectors.

Example: Some linear combinations of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are

Recall: $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

Example: $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the set of all vectors which can be written as __________________

____.

> **Theorem 4.1:** If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in a vector space V, then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a __________________ of V.
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> $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is called the __________________ **spanned** (or **generated**) by $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

Examples of Vector Spaces

Example. $\mathbb{R}^n$ for $n \geq 1$ with component-wise addition and scalar multiplication is a vector space.

Example. For $n \geq 0$, let

$$\mathbb{P}_n = \text{the set of all polynomials of degree at most } n$$

Elements of $\mathbb{P}_n$ have the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

where the coefficients $a_0, a_1, \ldots, a_n$ and the variable t are real numbers.

$\mathbb{P}_n$ with point-wise (like-term) addition and scalar multiplication is a vector space.

Examples of Subspaces

Example. $H = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

Important Note: The vector space $\mathbb{R}^2$ is **not** a subspace of $\mathbb{R}^3$ because $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$. (Vectors in $\mathbb{R}^3$ have three entries, and vectors in $\mathbb{R}^2$ have only two entries, so we cannot relate the two spaces.)