Learning Objective

- Understand how to find the determinant of an $n \times n$ matrix

Introduction to Determinants

Notation: Let A be an $n \times n$ matrix. Then A_{ij} is the $(n-1) \times (n-1)$ submatrix formed by deleting the ith row and jth column of A.

Example: Find the submatrix A_{23} obtained from the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \end{bmatrix} \qquad A_{23} =$$

Definition: For $n \geq 2$, the __________________________ of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$=$$

Example: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$.

Notation: Given $A = [a_{ij}]$, the __________________________ of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Using this new notation, we can rewrite the definition of the determinant as

$$\det A =$$

which we call the _______________________________________ **across the first row of** A.

Theorem 3.1: The determinant of an $n \times n$ matrix $A = [a_{ij}]$ can be computed by a cofactor expansion ____:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Remark: We can use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ by using a cofactor expansion **<u>down column 3</u>**.

Example: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 5 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

(*Hint:* At each step, choose a row or column that involves the least amount of computation.)

Theorem 3.2: If A is a __________________________ matrix, then $\det A$ is the product of the entries of the main diagonal of A.

Example: Compute the determinant of $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.