Learning Objectives

- Use the Invertible Matrix Theorem to connect properties of square matrices
- Determine whether a linear transformation is invertible

Characterizations of Invertible Matrices

Theorem 2.8: The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

(a) A is an ____________________ matrix.

(b) A is row equivalent to the ___________________________ matrix.

(c) A has _______ pivot positions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has __________________________________ solution.

(e) The columns of A form a linearly ___________________________ set.

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is ____________________.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has ___________________________ solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A _____________ $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps ___________________________.

(j) There is an $n \times n$ matrix C such that _________________.

(k) There is an $n \times n$ matrix D such that _________________.

(l) _______ is an invertible matrix.

Remarks:

1. The Invertible Matrix Theorem only applies to _________________ matrices.
2. The Invertible Matrix Theorem divides the set of all __________ matrices into two disjoint classes. Therefore, it also characterizes ________________________ (or singular) matrices.

Example: Use the Invertible Matrix Theorem to determine if the following matrices are invertible. (You should not have to do any row operations.) Be sure you can justify your answers.

a. $\begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix}$

b. $\begin{bmatrix} -2 & 3 & 1 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & -9 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 1 & 8 & 2 & 0 \\ -2 & -1 & -1 & 5 & 0 \\ 1 & 3 & -3 & 3 & 0 \\ 0 & 7 & -10 & 3 & 0 \\ 5 & 5 & -3 & -1 & 0 \end{bmatrix}$

Example: Suppose A is a 5×5 matrix, and suppose there is a vector $\mathbf{v}$ in $\mathbb{R}^5$ which is not a linear combination of the columns of A. What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$?

Invertible Linear Transformations

If A is an invertible $n \times n$ matrix, then for all $\mathbf{x}$ in $\mathbb{R}^n$, we have

$A^{-1}A\mathbf{x} =$ and $AA^{-1}\mathbf{x} =$

Definition: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be ____________________ if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

________________________ for all $\mathbf{x}$ in $\mathbb{R}^n$

________________________ for all $\mathbf{x}$ in $\mathbb{R}^n$

We call S the _________________ of T and write it as _______.

Theorem 2.9: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is __________________ if and only if A is an __________________ matrix. In that case, the linear transformation S given by $S(\mathbf{x}) =$ ______________ is the unique function satisfying the equations $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Example: Prove the following statement or provide a counterexample: $T : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one linear transformation if and only if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an onto linear transformation.