Learning Objectives

- Understand what it means for a matrix to be invertible
- Determine whether a 2×2 matrix is invertible
- Understand how to find the inverse of a matrix

The Inverse of a Matrix

The multiplicative inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = \frac{1}{7}$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

Definition: An $n \times n$ matrix A is said to be ____________________ if there exists an $n \times n$ matrix C such that

______________ and ______________

where $I = I_n$ is the $n \times n$ identity matrix. We call C the ____________________ of A.

FACT: If A is invertible, then the inverse of A is unique.

Notation: The unique inverse of A is usually denoted by A^{-1}, so that

__________ $= I$ and __________ $= I$

WARNING: Not all $n \times n$ matrices are invertible. Matrices that are *not* invertible are sometimes called **singular**, and invertible matrices are called **nonsingular**.

Theorem 2.4: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} =$$

If $ad - bc = 0$, then A is __________________________________.

The quantity $ad - bc$ is called the ___________________________ of A, and we write

$$\det A =$$

Theorem 2.4 says that a 2×2 matrix A is invertible if and only if __________ $\neq 0$.

Theorem 2.5: If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution ____________________.

Proof of Theorem 2.5: Suppose A is an invertible $n \times n$ matrix, and let $\mathbf{b}$ be any vector in $\mathbb{R}^n$. Then the vector __________ is a solution to the equation $A\mathbf{x} = \mathbf{b}$ since

$$A\mathbf{x} =$$

We can prove that this is the *unique* solution since if $\mathbf{u}$ is another solution to $A\mathbf{x} = \mathbf{b}$, we can left multiply both sides by _______ and get

____________________, ____________________ and ____________________

Example: Show that the matrix $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ is invertible. Then use the inverse of A to solve the system of linear equations

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$$

Theorem 2.6:

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = \_\_\_\_\_\_\_$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is

$$(AB)^{-1} = \_\_\_\_\_\_\_\_\_\_\_\_\_$$

c. If A is an invertible matrix, then so is A^T and

$$(A^T)^{-1} = \_\_\_\_\_\_\_\_\_\_\_\_\_$$

Proof of Theorem 2.6.b: Suppose A and B are $n \times n$ invertible matrices. Then

$(AB)(B^{-1}A^{-1}) =$

Similarly, $(B^{-1}A^{-1})(AB) =$

Remark: Theorem 2.6.b can be generalized to three or more invertible matrices. If A, B, and C are $n \times n$ invertible matrices, then

$$(ABC)^{-1} =$$

In general, the product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the ____________________ order.

Invertible $n \times n$ Matrices

We have a method for finding the inverse of a 2×2 invertible matrix. How do we find the inverse of an $n \times n$ invertible matrix for $n > 2$?

Theorem 2.7: An $n \times n$ matrix A is invertible if and only if A is row equivalent to _____, and in this case, any sequence of elementary row operations that reduces A to _____ also transforms _____ into A^{-1}.

An Algorithm for Finding A^{-1}

Algorithm for Finding A^{-1}

Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} \quad & \quad \end{bmatrix}$. Otherwise A does not have an inverse.

Example: Use the algorithm above to find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.