Learning Objectives

- Use matrix algebra to solve equations involving matrices
- Understand how to find the transpose of a matrix

Matrix Operations

Two ways to denote an $m \times n$ matrix A:

1. In terms of the *columns* of A: $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$
2. In terms of the *entries* of A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}$$

The **main diagonal entries** of A are ____________________.

Theorem 2.1: Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

a. $A + B =$ ______________

b. $(A + B) + C =$ ____________________

c. $A + 0 =$ _______

d. $r(A + B) =$ ____________________

e. $(r + s)A =$ ____________________

f. $r(sA) =$ ______________

Matrix Multiplication

Definition: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the ______________ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$. That is,

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} =$$

Example: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} =$$

$$A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -7 \end{bmatrix} =$$

So, $AB =$

> **Remark:** Each column of AB is a __________________________________ of the columns of A using weights from the corresponding columns of B.

The number of ________________ of A must match the number of __________ in B in order for a linear combination such as $A\mathbf{b}_1$ to be defined. Thus, if A is an $m \times n$ matrix and B is a $n \times p$ matrix, AB is a _____________ matrix.

Example: If A is a 4×3 matrix and B is a 3×2 matrix, what are the sizes of AB and BA, if they are defined?

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} =$$

BA would be $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$, which is __________________________.

> **The Row-Column Rule for Computing AB**
>
> If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of the corresponding entries from _____________ of A and ____________________ of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then
>
> $$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Example: Use the row-column rule to compute the following products:

a. $\begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix} =$

b. $\begin{bmatrix} 1 & 0 & -5 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & -4 & 1 \\ -1 & 0 & -2 \end{bmatrix} =$

Theorem 2.2: Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) =$ ____________ (associative law of multiplication)

b. $A(B + C) =$ _______________ (left distributive law)

c. $(B + C)A =$ _______________ (right distributive law)

d. $r(AB) =$ _________ $=$ _________
for any scalar r

e. $I_m A =$ ______ $= AI_n$ (identity for matrix multiplication)

WARNINGS:

The properties above are analogous to properties of multiplication of real numbers. But, NOT ALL real number properties correspond to matrix properties.

1. In general, $AB \neq$ ______. (See Example 7 on page 100.)
2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is __________________ in general that $B = C$. (See Exercise 10 on page 102.)
3. If a product AB is the zero matrix, you *cannot* conclude in general that either ____________ or ____________. (See Exercise 12 on page 103.)

Powers of a Matrix

If A is an $n \times n$ matrix and k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

The Transpose of a Matrix

Definition: If A is an $m \times n$ matrix, the ______________________ of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Example: If $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{bmatrix}$, then $A^T =$

Example: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute AB, $(AB)^T$, $A^T B^T$, and $B^T A^T$.

Theorem 2.3: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T =$ ______

b. $(A + B)^T =$ _____________

c. For any scalar r, $(rA)^T =$ __________

d. $(AB)^T =$ _____________