Learning Objectives

- Understand how to find the standard matrix of a linear transformation
- Determine whether a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one
- Determine whether a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$

The Matrix of a Linear Transformation

Example: Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation that maps $\mathbf{e}_1$ to $\mathbf{y}_1$ and $\mathbf{e}_2$ to $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ under T.

Solution: Observe that any vector $\mathbf{x}$ in $\mathbb{R}^2$ can be written as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{\quad\quad} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underline{\quad\quad} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{\quad\quad}\ \mathbf{e}_1 + \underline{\quad\quad}\ \mathbf{e}_2$$

Then since T is a **linear** transformation,

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \\ &= T(x_1\mathbf{e}_1) + T(x_2\mathbf{e}_2) \\ &= \underline{\quad\quad}\ T(\mathbf{e}_1) + \underline{\quad\quad}\ T(\mathbf{e}_2) \\ &= \underline{\quad\quad}\ \mathbf{y}_1 + \underline{\quad\quad}\ \mathbf{y}_2 \\ &= \begin{bmatrix} \quad\quad\quad \\ \\ \\ \end{bmatrix} \end{aligned}$$

Therefore, $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) =$

The previous example illustrates that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by what it does to the ________________ of the $n \times n$ identity matrix I_n.

Theorem 1.10: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = \underline{\hspace{2cm}} \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

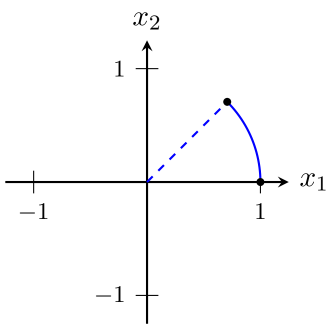
In fact, A is the $m \times n$ matrix whose jth column is the vector __________, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

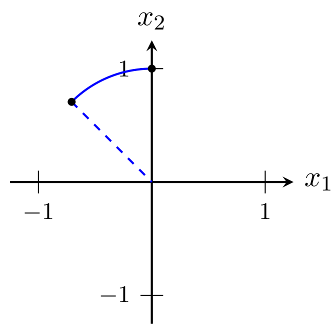
$$A = \begin{bmatrix} \qquad\qquad\qquad\qquad \end{bmatrix}$$

Definition: The matrix A above is called the ____________________ **matrix for the linear transformation** T.

Example: Find the standard matrix A for the projection of $\mathbb{R}^3$ onto the x_1x_2-plane $T(\mathbf{x}) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$.

Example: Find the standard matrix A for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates the plane about the origin through an angle of $\frac{\pi}{4}$ radians counterclockwise. *Hint:* Determine how T transforms $\mathbf{e}_1$ and $\mathbf{e}_2$. Knowledge of the unit circle could help.





Existence and Uniqueness Questions

Definition: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of ______________________ $\mathbf{x}$ in $\mathbb{R}^n$.

Equivalently, T is **onto** $\mathbb{R}^m$ when the range of T is all of the codomain $\mathbb{R}^m$. That is, T maps $\mathbb{R}^n$ **onto** $\mathbb{R}^m$ if for each $\mathbf{b}$ in the codomain $\mathbb{R}^m$ there exists at least one solution of ____________. Therefore, the question: "Does T map $\mathbb{R}^n$ **onto** $\mathbb{R}^m$?" is an ___________________ question.

Definition: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of ______________________ $\mathbf{x}$ in $\mathbb{R}^n$.

Equivalently, T is **one-to-one** if, for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a ____________ solution or ___________________. Therefore, the question: "Is T **one-to-one**?" is a ___________________ question.

Theorem 1.11: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has ____.

Theorem 1.12: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A _________________________.

b. T is one-to-one if and only if the columns of A ________________________________.

Example: Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \\ 0 & -1 & 4 \end{bmatrix}$$

Does T map $\mathbb{R}^3$ onto $\mathbb{R}^4$? Is T a one-to-one mapping?

Example: Let $T(x_1, x_2, x_3) = (3x_1 + x_2 - x_3, 6x_1 + 7x_2 + x_3)$.

a. Find the standard matrix of T.

b. Is T a one-to-one mapping?

c. Does T map $\mathbb{R}^3$ onto $\mathbb{R}^2$?