Learning Objectives

- Determine whether a specified vector is in the range of a matrix transformation
- Determine whether a transformation is linear

Introduction to Linear Transformations

It turns out that a matrix equation $A\mathbf{x} = \mathbf{b}$ can arise in linear algebra (and in its applications) in a way that is not directly connected with linear combinations of vectors.

> We can think of the matrix A as an object that "acts" on a vector $\mathbf{x}$ by multiplication to produce a new vector called $A\mathbf{x}$.

Example: $\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ -12 \\ -4 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$

If A is an $m \times n$ matrix, then solving $A\mathbf{x} = \mathbf{b}$ amounts to finding all ________________ in $\mathbb{R}^n$ which are transformed into the vector $\mathbf{b}$ in ______ under the "action" of multiplication by A.

> **Definition:** A ________________________ (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

The set $\mathbb{R}^n$ is called the ____________ of T and $\mathbb{R}^m$ is called the _______________ of T.

For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the ____________ of $\mathbf{x}$.

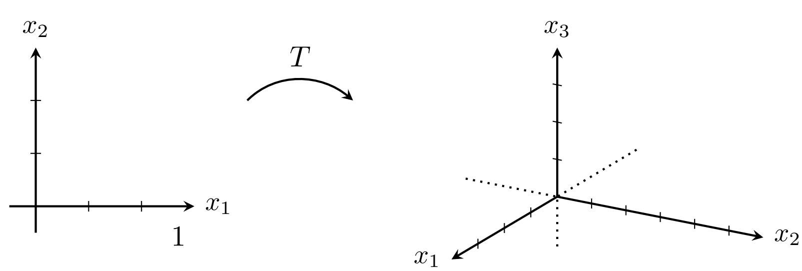
The set of all images $T(\mathbf{x})$ is called the ____________ of T.

Matrix Transformations

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define a transformation $T :$ ____ $\rightarrow$ ____ by $T(\mathbf{x}) = A\mathbf{x}$.

Describe the image of $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ under the transformation T.

$T(\mathbf{u}) = A\mathbf{u} =$



Example: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Define a transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Find an $\mathbf{x}$ in $\mathbb{R}^3$ whose image under T is $\mathbf{b}$.

b. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$? (*uniqueness problem*)

c. Determine if $\mathbf{c}$ is in the range of the transformation T. (*existence problem*)

Linear Transformations

Definition: A transformation T is ____________ if

(i) $T(\mathbf{u}+\mathbf{v}) =$ __________________ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;

(ii) $T(c\mathbf{u}) =$ ____________ for all scalars c and all $\mathbf{u}$ in the domain of T.

Recall Theorem 1.5 says that if A is an $m \times n$ matrix, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ has the properties

$$A(\mathbf{u}+\mathbf{v}) = \underline{\hspace{3cm}} \qquad \text{and} \qquad A(c\mathbf{u}) = \underline{\hspace{2cm}}$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and all scalars c.

Thus, we have the following important fact:

Every _______________ transformation is a _______________ transformation.

The definition of a linear transformation implies the following properties:

If T is a linear transformation, then

$$T(\mathbf{0}) = \underline{\hspace{1.5cm}}$$

and

$$T(c\mathbf{u}+d\mathbf{v}) = \underline{\hspace{2.5cm}} + \underline{\hspace{2.5cm}}$$

for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d.

Note that if a transformation satisfies the second property above, then it must be linear.

Repeated application of this property gives us a useful generalization, referred to as the **superposition principle**:

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = \underline{\hspace{6cm}}$$

Example: Define a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$.

Show that T is a linear transformation.

Example: Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(\mathbf{x}) = \begin{bmatrix} |x_1 + x_3| \\ 2 + 5x_2 \end{bmatrix}$.

Show that T is **not** a linear transformation.