Learning Objectives

- Determine whether a set of vectors is linearly independent
- Determine whether the columns of an $m \times n$ matrix are linearly independent

Linear independence

> **Definition:** A set of vectors $S = \{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ is ________________________ if, whenever $c_1\mathbf{v_1} + \cdots + c_t\mathbf{v_t} = \mathbf{0}$, we have $c_1 = \cdots = c_t = 0$.
>
> A set of vectors is $S = \{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ is linearly dependent if there exist $c_1, \dots, c_t \in \mathbb{R}$ with some $c_i \neq 0$, such that ________________________

Example: Since $4\begin{bmatrix}1\\-1\\1\end{bmatrix} - 6\begin{bmatrix}0\\1\\0\end{bmatrix} + 2\begin{bmatrix}-2\\5\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, we conclude that the set

$$\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\5\\-2\end{bmatrix}\right\}$$

is linearly __________________.

> **Remark:** A set $S = \{\mathbf{v}\}$ with one vector is linearly dependent if and only if $\mathbf{v}$ is _____________.
>
> A set $S = \{\mathbf{v}, \mathbf{w}\}$ with two vectors is linearly dependent if and only if one of the vectors is a ________________________ of the other.

Example: Do the vectors $\mathbf{v} = \begin{bmatrix}3\\-1\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}2\\-3\end{bmatrix}$ for a linearly independent set?

Linear independence and matrices

> **Theorem** For a matrix $A = \begin{bmatrix}\mathbf{v_1} & \cdots & \mathbf{v_t}\end{bmatrix}$, the columns of A form a linearly independent set if and only if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a __________________ solution.

Example: Consider the vectors

$$\begin{bmatrix}0\\1\\0\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\\4\\5\end{bmatrix}, \begin{bmatrix}0\\0\\2\\-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\2\\1\\-1\\1\end{bmatrix} \in \mathbb{R}^5.$$

To test whether they form a linearly independent set, we put them into a matrix

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 2 & 4 & -1 & -1 \\ 3 & 5 & -1 & 1 \end{bmatrix}$$

and row reduce to echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are no free variables, we conclude that ______________________________________.

> **Theorem** Any set $\{\gtrapprox_{\nVdash}, \ldots, \gtrapprox_{\approx}\} \subseteq \mathbb{R}^n$ with t _____ n is linearly dependent.

Linear independence geometrically

> **Theorem** A set of (at least two) vectors is linearly dependent if and only if one of the vectors is the set is a _________________________of the rest.

> **Theorem** A set $S = \{\mathbf{v_1}, \ldots, \mathbf{v_t}\} \in \mathbb{R}^n$ is linearly independent if and only if
>
> - $\mathbf{v_1} \neq \mathbf{0}$, and
> - $\mathbf{v_2} \notin \text{Span}\{\mathbf{v_1}\}$, and
>
> $\vdots \quad \vdots \quad \vdots$
> - $\mathbf{v_t} \neq \text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_{t-1}}\}$.