Learning Objectives

- Determine when a homogeneous system has a nontrivial solution
- Understand how to write a general solution in parametric vector form
- Understand how to write the equation of a line or plane in parametric vector form
- Understand how the solution set of $A\mathbf{x} = \mathbf{b}$ relates to the solution set of $A\mathbf{x} = \mathbf{0}$

Homogeneous Linear Systems

Definition: A system of linear equations is said to be ________________________ if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Example: Write the following homogeneous system as a matrix equation.

$$\begin{aligned} x_1 + 10x_2 &= 0 \\ 2x_1 + 20x_2 &= 0 \end{aligned}$$

Without solving the system, decide whether the system has a solution in $\mathbb{R}^2$. That is, does there exist a vector $\mathbf{x}$ in $\mathbb{R}^2$ that satisfies the homogenous matrix equation?

Remark: A homogeneous system $A\mathbf{x} = \mathbf{0}$ ***always*** has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the _____________ **solution**.

For a given equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a __________________ **solution**, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$.

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one ________________________.

Example: Solve the homogeneous system:

$$\begin{aligned} 3x_1 + 6x_2 &= 0 \\ 2x_1 + 4x_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} 3 & 6 & 0 \\ 2 & 4 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

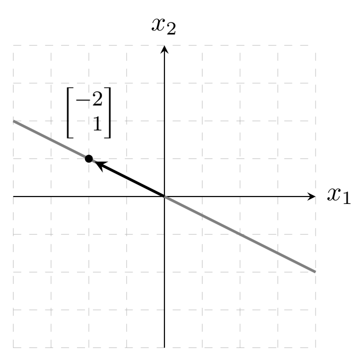
Solution set: $\begin{cases} x_1 = \\ x_2 \quad \underline{\hspace{3cm}} \end{cases}$

We can also write our solution as a vector:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

This is called the **parametric vector form** of the solution.

We can also graph our solution set:



Geometrically, the solution set is a $\underline{\hspace{3cm}}$ through $\mathbf{0}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$.

Solutions of Nonhomogeneous Systems

Example: Solve the non-homogeneous system:

$$\begin{aligned} 3x_1 + 6x_2 &= 9 \\ 2x_1 + 4x_2 &= 6 \end{aligned}$$

Notice that the coefficient matrix of this system is the same as the one in the example above.

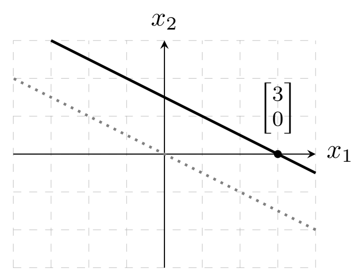
$$\begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution set: $\begin{cases} x_1 = \\ x_2 \quad \underline{\hspace{3cm}} \end{cases}$

Parametric Vector Form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 - 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} + x_2 \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Graph:



Geometrically, the solution set is a line in $\mathbb{R}^2$ that passes through the point $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and is ______________________ to the line through $\mathbf{0}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

> **Remark:** Notice from the two examples above that the solution set of $A\mathbf{x} = \mathbf{b}$ is ______________________ to the solution set of $A\mathbf{x} =$ ______.

The following theorem says that when a nonhomogeneous system is consistent, we can find its solution(s) from the corresponding homogeneous system:

> **Theorem 1.6:** Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form
>
> $$\mathbf{w} =$$
>
> where ______ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Warning: Theorem 1.6 applies only to an equation $A\mathbf{x} = \mathbf{b}$ that has at least one ________________ solution. Otherwise, the solution set is ________________.