Learning Objectives

- Understand what it means for a set of vectors to span $\mathbb{R}^m$
- Determine whether the columns of an $m \times n$ matrix span $\mathbb{R}^m$
- Understand how to compute $A\mathbf{x}$ for an $m \times n$ matrix A and a vector $\mathbf{x}$ in $\mathbb{R}^n$
- Use linearity of matrix multiplication to compute $A(\mathbf{u} + \mathbf{v})$ or $A(c\mathbf{u})$

The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Definition: If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and x is a vector in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

Remark: $A\mathbf{x}$ is defined only if the number of _______________ of A equals the number of entries in $\mathbf{x}$.

Example: $\begin{bmatrix} 2 & 1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \\ 3 \end{bmatrix} =$

Example: Consider the following system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 1x_3 &= 0 \\ -2x_1 + 5x_2 + 4x_3 &= 1 \\ x_1 \qquad + 2x_3 &= -1 \end{aligned}$$

First, write the system as a vector equation involving a linear combination of vectors.

Now, write the system as a product of a matrix and a vector.

We now have three ways to view solving a system of linear equations:

Theorem 1.3: If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and if $\mathbf{b}$ is a vector in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

Existence of Solutions

Remark: Our definition of matrix multiplication leads to the following useful fact.

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a ____________________________ of the _______________ of A.

Question: Let A be an $m \times n$ matrix. Does $A\mathbf{x} = \mathbf{b}$ have a solution for all possible $\mathbf{b}$?

Example: Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible $\mathbf{b}$?

Strategy: Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{bmatrix}$$

Conclusion:

Definition: We say **the columns of** $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p \end{bmatrix}$ **span** $\mathbb{R}^m$ if ____________ vector $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_p$. That is, the columns of A span $\mathbb{R}^m$ if

$$\text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_p\} =$$

Theorem 1.4: Let A be an $m \times n$ matrix. Then the following statements are equivalent. That is, for a particular A, they are either all true or all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation ______________ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a ____ of the columns of A.

c. The columns of A ______________ $\mathbb{R}^m$.

d. A has a ___________________________ in every row.

Warning: Theorem 1.4 is about a **coefficient matrix** A, not an augmented matrix.

Example: Do the columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix}$ span $\mathbb{R}^3$?

Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the i^{th} entry of $A\mathbf{x}$ is the dot product of row i of A and the vector $\mathbf{x}$.

Example: Dot product: $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1(1) + 2(2) - 1(3) = 2$

Example: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then

$$A\mathbf{x} = \begin{bmatrix} (\text{row 1 of } A) \cdot \mathbf{x} \\ (\text{row 2 of } A) \cdot \mathbf{x} \\ (\text{row 3 of } A) \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1(1) + 2(2) - 1(3) \\ -3(1) - 4(2) + 2(3) \\ 5(1) + 2(2) + 3(3) \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

> **Definition:** The _______________ **matrix**, denoted I_n, is an $n \times n$ matrix with 1's along the diagonal and zeros everywhere else.

Example: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $I_4 = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$

Properties of the Matrix-Vector Product $A\mathbf{x}$

> **Theorem 1.5:** If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:
>
> a. $A(\mathbf{u} + \mathbf{v}) =$ _______________
>
> b. $A(c\mathbf{u}) =$ _________

Example: Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

Verify Theorem 5(a) in this case by computing $A(\mathbf{u} + \mathbf{v})$ and $A\mathbf{u} + A\mathbf{v}$.