Learning Objectives

- Determine whether a vector is in $\mathbb{R}^2, \mathbb{R}^3$, or $\mathbb{R}^n$ for some integer $n > 0$
- Understand how to scale and add vectors
- Determine if a vector can be written as a linear combination of speci ed vectors
- Determine if a vector is in a subset spanned by specified vectors
- Understand how to describe the span of a set of vectors

Vectors in $\mathbb{R}^2$

> **Definition:**
> - A ________________is a matrix with one column.
> - The set of all vectors with two entries is denoted ______.

Remark: Given two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and a constant (*scalar*) $c \in \mathbb{R}$, we have

- Equality: $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = u_2$ and $v_1 = v_2$;
- Addition: $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$;
- Scaling: $c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$.

Example: For $\mathbf{u} = \begin{bmatrix} 5 \\ -1/2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2/3 \\ 6 \end{bmatrix}$, find $4\mathbf{u} - 6\mathbf{v}$.

Geometric Descriptions of $\mathbb{R}^2$

Consider the usual Cartesian rectangular coordinate system on the plane. Because each point in the plane is determined by an ordered pair of numbers, we can identify a geometric point (a, b) with the vector $\begin{bmatrix} a \\ b \end{bmatrix}$. We often visualize vectors in $\mathbb{R}^2$ as arrows in the plane. When we do this, the points on the stalk of the arrow have no special significance, just the endpoint.

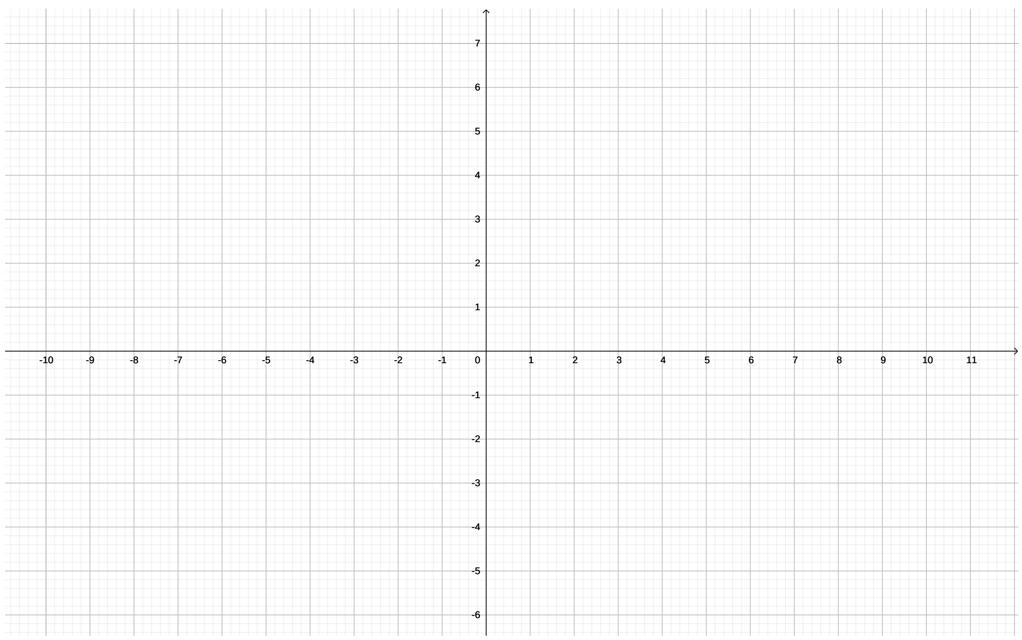
Often it is preferable to think in terms of arrows or directions, for example, when we want to think of adding vectors. Other times it makes more sense to think of vectors as points or destinations, for example, when we are considering solutions of linear systems.

> **Parallelogram rule for addition:** The vector $\mathbf{u} + \mathbf{v}$ is the fourth corner of the parallelogram with corners $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$.

We can also think of $\mathbf{u} + \mathbf{v}$ as the vector obtained by moving the arrow $\mathbf{v}$ so that its tail starts at the end of $\mathbf{u}$.

Scaling: If we scale $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ by c then we get the vector corresponding to the point (cu_1, cu_2), which is a point on the line formed by $(0,0)$ and (u_1, u_2) with length $|c|$ times that of $\mathbf{u}$, and opposite direction if c is negative.

Example: Draw the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ as arrows below. Use the parallelogram rule to draw $\mathbf{u} + \mathbf{v}$, and draw $\frac{-1}{2}\mathbf{v}$ using the geometric description above.



Vectors in $\mathbb{R}^n$

Definition: $\mathbb{R}^n$ is the set of all ________________________________.The vector in $\mathbb{R}^n$ all of whose entries are zero is called the ____________________and denoted $\nvdash$.

Remark: Addition and scalar multiplication in $\mathbb{R}^n$ are defined the same way as in $\mathbb{R}^2$ above.

General algebraic properties of vector operations

For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and all scalars $c, d \in \mathbb{R}$, we have

1. $\mathbf{u} + \mathbf{v} =$_____________
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} =$_____________
3. $\mathbf{u} + \mathbf{0} =$_____________
4. $\mathbf{u} + (-\mathbf{u}) =$_____________
5. $c(\mathbf{u} + \mathbf{v}) =$_____________
6. $(c + d)\mathbf{u} =$_____________
7. $c(d\mathbf{u}) =$_____________
8. $1\mathbf{u} =$_____________

Linear combinations

> **Definition:** Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$, the vector
> $$\mathbf{b} = a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m$$
> is called the ______________________________ of $\mathbf{v}_1, \ldots, \mathbf{v}_m$ with *weights* $a_1, \ldots, a_m$.

Example: Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}.$$

We claim that $\mathbf{b} = \begin{bmatrix} 7 \\ -14 \\ 0 \end{bmatrix}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. Prove this by computing $2\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_3$.

Example: Determine if $\mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ as in the last example. To do this, we need to find weights x_1, x_2, and x_3 such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{c}$:

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ -2x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -6x_3 \\ 8x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 5x_3 \\ -2x_1 + x_2 - 6x_3 \\ 2x_2 + 8x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

Thus, we need to determine whether the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & + & & & 5x_3 & = & 2 \\ -2x_1 & + & x_2 & - & 6x_3 & = & -1 \\ & & 2x_2 & + & 8x_3 & = & 6 \end{array}$$

is consistent. Do this by writing out the augmented matrix of the system and converting it to row echelon form to find the general solution.

> **Definition:** A ______________________________ is an equation of the form
>
> $$x_1\mathbf{v_1} + \cdots + x_n\mathbf{v_n} = \mathbf{b}$$
>
> where $\mathbf{v_1}, \ldots, \mathbf{v_n}, \mathbf{b}$ are vectors. Its solution set is the same as that of the system corresponding to the augmented matrix $[\mathbf{v_1} \cdots \mathbf{v_n}\ \mathbf{b}]$. In particular, this system is consistent if and only if $\mathbf{b}$ is a ______________________________ of $\mathbf{v_1}, \ldots, \mathbf{v_n}$.

Span

> **Definition:** The span of a set of vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ is the set of vectors that are ____________________________________.

The span of *one* nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$ is the set of scalar multiples of $\mathbf{v}$: Span$\{\mathbf{v}\}$ is just the line through $\mathbf{0}$ and $\mathbf{v}$.
If $\mathbf{v}$ and $\mathbf{w}$ are *two* nonzero vectors in $\mathbb{R}^3$, and $\mathbf{w}$ is not a scalar multiple of $\mathbf{v}$, then Span$\{\mathbf{v}, \mathbf{w}\}$ is the plane through $\mathbf{0}, \mathbf{v}$, and $\mathbf{w}$.

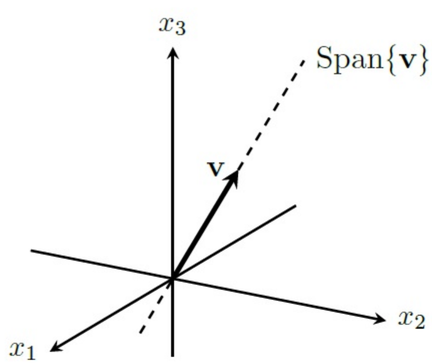


Figure 1: Span$\{\mathbf{v}\}$ as a line through the origin

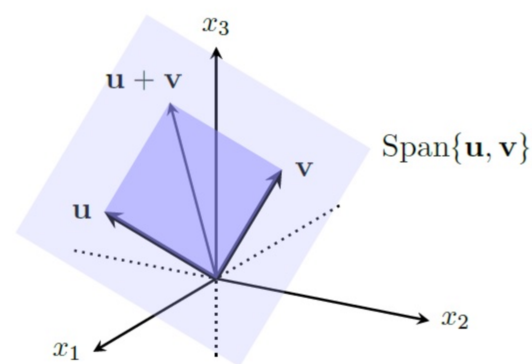


Figure 2: Span$\{\mathbf{u}, \mathbf{v}\}$ as a plane through the origin

Example: We can rephrase the conclusion of the previous example as saying that ______ is/is not (choose one) in the span of ____________.