Learning Objectives

- Use row operations to find a row echelon form of a matrix
- Use row operations to find the reduced row echelon form of a matrix
- Determine where the pivot positions are in a matrix
- Understand the difference between basic variables and free variables
- Describe the existence or uniqueness of solutions in terms of pivot positions
- Determine values of parameters that make a solution unique

Row Reduction and Echelon Forms

Definition:

- A ________________ row or column in a matrix means a row or column contains at least one nonzero entry.
- A __________________________ of a row refers to the leftmost nonzero entry (in a nonzero row).

Definition: A matrix is in _____________________________ (REF) if it has the following three properties:

1. All ________________ rows are above any rows of all zeros.
2. Each __________________________ of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are _____________.

If a matrix in row echelon form satisfies the following additional conditions, then it is in ____ (RREF).

4. The leading entry in each nonzero row is _______.
5. Each leading _______ is the only nonzero entry in its column.

Example: Consider the matrices

$$\begin{bmatrix} 5 & -1 & -7 & 0 & 2 \\ 0 & -4 & -1 & 3 & 9 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -3 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

These matrices are both in ______________________ form.
The second matrix is in ____________________________________ form.

Theorem 1.1: Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one ________________________________ matrix.

Pivot Positions

Definition:

- A ____________ **position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced row echelon form of A.
- A ____________ **column** is a column of A that contains a pivot position.

Example: Identify the pivot positions and pivot columns in the previous example.

The Row Reduction Algorithm

This algorithm finds the reduced row echelon form of a matrix.

1. Begin with the leftmost nonzero column. The top position in this column is a pivot position.
2. If necessary, interchange rows so that a nonzero entry is in the pivot position.
3. Use row operations to create zeros in all positions below the pivot.
4. Ignore the row with the pivot and all rows above it, and apply steps 1−3 to the remaining submatrix. Repeat the process until there are no more nonzero rows.
5. Starting at the rightmost pivot and working up and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by scaling the row.

Example: Apply elementary row operations to transform the following matrix first into row echelon from and then into reduced row echelon form:

$$\begin{bmatrix} 0 & 2 & -8 & 8 \\ 1 & 3 & -4 & 5 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

Solutions of Linear Systems

> **Definition:** A ____________ **variable** corresponds to a pivot column of the augmented matrix, and a ____________ **variable** corresponds to a non-pivot column.

Example: Solve the following linear system

$$\begin{aligned} x_1 - 5x_2 \qquad &= -4 \\ -2x_1 + 10x_2 + x_3 &= 10 \\ -x_1 + 5x_2 + 3x_3 &= 10 \end{aligned}$$

First find the augmented matrix: $\begin{bmatrix} 1 & -5 & 0 & -4 \\ -2 & 10 & 1 & 10 \\ -1 & 5 & 3 & 10 \end{bmatrix}$

Then use the row reduction algorithm to find the RREF: $\begin{bmatrix} 1 & -5 & 0 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basic variables: ____________

Free variable: _________

The general solution:

$$\begin{cases} x_1 = \\ x_2 \text{ is } \_\_\_\_\_\_\_\_\_\_\_\_ \\ x_3 = \end{cases}$$

> **Remark:** Since the linear system in the last example has a ____________ **variable** in its solution set, there are ____________________________ solutions to this system of linear equations.
>
> Since x_2 is a ____________ variable, we can choose any value for x_2 and then use the equation we found above to solve for x_1.
>
> For example, when $x_2 = 0$, the solution is ____________.
>
> When $x_2 = 1$, the solution is ____________.
>
> *Each different choice of x_2 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_2.*

Existence and Uniqueness Questions

Theorem 1.2: Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a ______________ column.

In other words, a linear system is consistent if and only if an echelon form of the augmented matrix has **no** row of the form

$$[\qquad\qquad\qquad\qquad] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either

1. a _________________ solution when there are no free variables
2. ___________________________________ solutions when there is at least one free variable.

Example: The **augmented** matrices below are in row echelon form. Determine whether the corresponding linear systems have **one unique** solution, **infinitely many** solutions, or **no** solutions.

a.
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 9 & 8 & 7 & 6 & 5 & 4 \\ 0 & 3 & 2 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

c.
$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 7 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d.
$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 7 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$

e.
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 \end{bmatrix}$$

f.
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

g.
$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 7 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

h.
$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 7 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$