Learning Objectives

- Understand what is a system of linear equations
- Use matrix notation and augmented matrices to rewrite systems of linear equations
- Recognize and use elementary row operations to find equivalent linear systems

Systems of linear equations

Definition:

- A ________________ equation is an equation that can be written in the form

$$a_1x_1 + \cdots + a_nx_n = b,$$

where $a_1, \ldots, a_n, b$ are constants, and $x_1, \ldots, x_n$ are the variables.

- A __________________________ is a collection of one or more linear equations in the same set of variables.
- A ________________ of a system of linear equations is an ordered list of numbers $(s_1, \ldots, s_n)$ that makes each equation true when we substitute $x_1 = s_1, \ldots, x_n = s_n$.
- Two systems of linear equations are *equivalent* if they have the same set of ________________.
- A system of linear equations is ________________ if it has at least one solution.

Example: The two systems of equations

$$\begin{aligned} x &= 1 \\ y &= -1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + y &= 0 \\ x - y &= 2 \end{aligned}$$

are equivalent since, for each, the ___________________ consists of the (same) single element ________________.

The augmented matrix of a linear system

This is easiest to introduce in an example:

For the linear system

$$\begin{array}{ccccccc} 3x_1 & - & x_2 & & & = & -5 \\ \sqrt{2}x_1 & & & + & x_3 & = & 0 \\ x_1 & + & \pi x_2 & - & x_3 & = & 1, \end{array}$$

the ________________ is

$$\begin{bmatrix} 3 & -1 & 0 \\ \sqrt{2} & 0 & 1 \\ 1 & \pi & -1, \end{bmatrix},$$

and the _______________ is

$$\begin{bmatrix} 3 & -1 & 0 & -5 \\ \sqrt{2} & 0 & 1 & 0 \\ 1 & \pi & -1 & 1 \end{bmatrix}.$$

Note: Some people like to include an extra bar in the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & -5 \\ \sqrt{2} & 0 & 1 & 0 \\ 1 & \pi & -1 & 1, \end{array}\right]$$

since the last column is serving a different purpose than the other ones.

Warning: It is important to "line up" the coefficients for the same variable in the same column; that's why we included zeros above. Note that if different people choose to order their variables in a different way (e.g., I say x goes before y and you say y goes before x) then they will get different augmented matrices for the same system.

Example: To find the augmented matrix of the system

$$\begin{gathered} 2(1-x) = 4y - z \\ 7z - y = 1, \end{gathered}$$

first we clean up the system by "foil"ing out the coefficients, and moving the variable parts to one side and the constant parts to the other side. For any "missing" variables in an equation, leave a blank space so that all variables line up.

$$\begin{array}{rcrcrcr} -2x & - & 4y & + & z & = & -2 \\ & - & y & + & 7z & = & 1. \end{array}$$

Then we just box them up (you fill it in):

$$\begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{bmatrix}.$$

Elementary row operations

Definition:

- An *elementary row operation* on a matrix is any of the following three operations:
  - (_______________) Add to one row a multiple of another row.
  - (_______________) Interchange two rows.
  - (_______________) Multiply all entries in a row by a nonzero constant.
- Two matrices are _________________if there is a sequence of elementary row operations that transforms one matrix into the other.

Theorem If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example: Consider the linear system

$$\begin{aligned} 2a + 3c - d &= 7 \\ -b + c + d &= 2 \\ 2b + 9d &= 0. \end{aligned}$$

The augmented matrix of this system is

$$\begin{bmatrix} 2 & 0 & 3 & -1 & 7 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & 2 & -2 & 9 & 0 \end{bmatrix}.$$

We perform the replacement operation "add 2 times row two to row three" to get

$$\begin{bmatrix} 2 & 0 & 3 & -1 & 7 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 11 & 0 \end{bmatrix},$$

and then the scaling operation "multiply all entries in row three by 1/11" to get

$$\begin{bmatrix} 2 & 0 & 3 & -1 & 7 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Reinterpreting the last row as an equation, we see that $d = 0$ for any solution. Note that we could have done the exact same thing with equations rather than matrices ("add 2 times equation two to equation three" then "multiply equation three by 1/11"); a first advantage of dealing with augmented matrices is that we don't need to write the a, b, c, d's and $+$'s and $=$'s over and over.

In the next section, we will discuss an algorithm to use elementary row operations to transform the augmented matrix of a system into another augmented matrix for which the solution set is very easy to find.