Álgebra Conmutativa, Fall 2019, Homework #4

(1) Find systems of parameters for each of the following local rings:

(a) $R = \dfrac{\mathbb{Z}[x,y]_{(7,x,y)}}{(3x^2 - 49xy^2)}$.

(b) $R = \mathbb{C}[x^5, x^4y, xy^4, y^5]_{(x^5,x^4y,xy^4,y^5)}$.

(c) $R = \dfrac{\mathbb{C}[x,y,z]_{(x,y,z)}}{(x^3, x^2y, x^2z, xyz)}$.

(d) $R = \dfrac{\mathbb{C}[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]_{(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23})}}{(x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21}, x_{12}x_{23} - x_{13}x_{22})}$.

(2) Let K be a field, and R be a finitely generated graded K-algebra with $R_0 = K$. Let $y_1, \dots, y_t \in R$ be homogeneous. Show that $\{y_1, \dots, y_t\}$ is a homogeneous system of parameters of R if and only if $K[y_1, \dots, y_t]$ is a (graded) Noether normalization of R.

(3) An $n \times n$ magic square with row sum t is an $n \times n$ matrix with nonnegative integer entries such that every row and every column adds up to t. Show that, if n is fixed, the function

$$f_n(t) = \text{ number of } n \times n \text{ magic squares with row sum } t$$

agrees with a polynomial[1] for $t \gg 0$.

(4) Let K be a field. Let $\mathfrak{p} \subset R := K[x_1, \dots, x_m]$ and $\mathfrak{q} \subset S := K[y_1, \dots, y_n]$ be prime ideals. Let $T = R \otimes_K S = K[x_1, \dots, x_m, y_1, \dots, y_n]$. Show that $\dim(T/(\mathfrak{p}T + \mathfrak{q}T)) = \dim(R/\mathfrak{p}) + \dim(S/\mathfrak{q})$.

(5) Let K be a field, $R := K[x_1, \dots, x_n]$, $\mathfrak{m} = (x_1, \dots, x_n)$, and $\mathfrak{p}, \mathfrak{q} \subseteq \mathfrak{m}$ be prime ideals. Then $\text{height}(\mathfrak{p}) + \text{height}(\mathfrak{q}) \geq \text{height}(\mathfrak{p} + \mathfrak{q})$.[2]

(6) Let K be a field, and $R := K[x_1, x_2, x_3, x_4]/(x_1x_2 - x_3x_4)$. Find prime ideals $\mathfrak{p}, \mathfrak{q} \subset (x_1, x_2, x_3, x_4)$, such that $\text{height}(\mathfrak{p}) + \text{height}(\mathfrak{q}) < \text{height}(\mathfrak{p} + \mathfrak{q})$.

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[1] Hint: Consider the K-algebra $K[x_{11}^{a_{11}} x_{12}^{a_{12}} \cdots x_{nn}^{a_{nn}} \mid A = [a_{ij}] \text{ is a magic square}]$

[2] Hint: Let $S = K[y_1, \dots, y_n]$ and $\mathfrak{q}'$ be the same ideal as $\mathfrak{q}$ in the y variables. Use the fact that $R/(\mathfrak{p} + \mathfrak{q}) \cong (R \otimes_K S)/(\mathfrak{p} + \mathfrak{q}' + (x_1 - y_1, \dots, x_n - y_n))$.