Álgebra Conmutativa, Fall 2019, Homework #3

DEFINITION: A ring homomorphism $\phi : R \to S$ is *faithfully flat* if ϕ is flat and for every nonzero R-module M, the S-module $S \otimes_R M$ is nonzero.

(1) Show that, in each of the three cases specified below, ϕ is faithfully flat.
   (a) $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are local rings, and $\phi : R \to S$ is a flat homomorphism with $\phi(\mathfrak{m}) \subseteq \mathfrak{n}$.
   (b) $\alpha : R \to S$ is a faithfully flat map, T is an R-algebra, and $\phi = \alpha \otimes T : T \to S \otimes_R T$.
   (c) $K \subseteq L$ are fields, R is a K-algebra, and $\phi : R \to L \otimes_K R$ is the map $\phi(r) = 1 \otimes r$.

(2) Let $\phi : R \to S$ be faithfully flat.
   (a) Show[1] that ϕ is injective.
   (b) Show that ϕ^* is surjective.

(3) Show that any flat homomorphism satisfies the conclusion of the "going down" theorem.

(4) Each of the following rings is a domain. Find its dimension.
   (a) $\mathbb{C}[x^3y^3, x^3y^2z, x^3z^3] \subseteq \mathbb{C}[x, y, z]$.
   (b) $R = \dfrac{\mathbb{C}[x_1, x_2, x_3, x_4]}{(x_1x_3 - x_2^2, x_2x_4 - x_3^2, x_1x_4 - x_2x_3)}$.
   (c) $R = \dfrac{\mathbb{C}[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]}{(x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21}, x_{12}x_{23} - x_{13}x_{22})}$.

(5) Let $R = \mathbb{C}[x, y, z]$, and $I = (x^3, x^2y, x^2z, xyz)$. Find a prime filtration of R/I and a primary decomposition of I.

(6) Let R be a Noetherian ring, and I be an ideal. Consider a collection of minimal primary decompositions of I:
$$I = \mathfrak{q}_{1,\alpha} \cap \cdots \cap \mathfrak{q}_{s,\alpha}, \quad \alpha \in \Lambda$$
where, for each α, $\sqrt{\mathfrak{q}_{i,\alpha}} = \mathfrak{p}_i$.
   (a) Suppose that $\mathfrak{p}_j$ is not contained in any other associated prime of I, and let $W = R \smallsetminus \bigcup_{i \neq j} \mathfrak{p}_i$. Find some minimal primary decompositions of $I(W^{-1}R) \cap R$.
   (b) Show (by induction on s) that if we we take components $\mathfrak{q}_{1,\alpha_1}, \ldots, \mathfrak{q}_{s,\alpha_s}$ from different primary decompositions of I, that we can put them together to get a primary decomposition of I; namely $I = \mathfrak{q}_{1,\alpha_1} \cap \cdots \cap \mathfrak{q}_{s,\alpha_s}$.

[1] Hint: Show that for any ideal $J \subseteq R$, $J \otimes_R S \cong JS$, and let J be the kernel of ϕ.