Álgebra Conmutativa, Fall 2019, Homework #2

(1) Describe all of the elements of $\mathrm{Spec}(\mathbb{C}[x])$ and $\mathrm{Spec}(\mathbb{R}[x])$, and describe the map $\mathrm{Spec}(\mathbb{C}[x]) \to \mathrm{Spec}(\mathbb{R}[x])$ induced by the inclusion map $\mathbb{R}[x] \subseteq \mathbb{C}[x]$.

(2) Let K be an algebraically closed field. Let $R = K[x_1, \dots, x_m]/I$ and $S = K[y_1, \dots, y_n]/J$ be finitely generated K-algebras. Let $\varphi : R \to S$ be a K-algebra homomorphism; write $\varphi(x_i) = f_i(y_1, \dots, y_n)$ for some polynomial f_i. Recall that any $\mathfrak{n} \in \max(S)$ can be written as $\mathfrak{n} = \mathfrak{n}_{\underline{b}} := (y_1 - b_1, \dots, y_n - b_n)$ for some $\underline{b} = (b_1, \dots, b_n) \in Z_K(J)$, and similarly for $\mathfrak{m} \in \max(R)$. Thus, for the map on maximal ideals

$$\varphi^* : \max(S) \to \max(R) \qquad \varphi^*(\mathfrak{n}) = \varphi^{-1}(\mathfrak{n})$$

we can write $\varphi^*(\mathfrak{n}_{\underline{b}}) = \mathfrak{m}_{\underline{a}}$ for some $\underline{a} \in Z_K(I)$. Find a formula for $\underline{a}$ in terms of $\underline{b}$ and $f_1, \dots, f_m$.

(3) Let R be a ring of characteristic $p > 0$.
  (a) Show that the map $F : R \to R$ given by $F(r) = r^p$ is a ring homomorphism.
  (b) Show that F is module-finite if and only if it is algebra-finite.
  (c) Show that the map on spectra induced by F is the identity map.

(4) Let R be a finitely generated $\mathbb{Z}$-algebra, and $\mathfrak{m} \subseteq R$ a maximal ideal. Show that $R/\mathfrak{m}$ is finite.

(5) Let R be a ring and I an ideal. A *minimal prime* of I is a prime $\mathfrak{p}$ such that $I \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q}$ prime implies that $\mathfrak{q} = \mathfrak{p}$. We write $\mathrm{Min}(I)$ for the set of minimal primes of I. Show that $V(I) = \bigcup_{\mathfrak{p} \in \mathrm{Min}(I)} V(\mathfrak{p})$. Conclude that $\sqrt{I} = \bigcap_{\mathfrak{p} \in \mathrm{Min}(I)} \mathfrak{p}$.

(6) Let K be a field, and $\Phi_{\underline{f}} : K^n \to K^n$ be given by the rule

$$\Phi_{\underline{f}}(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))$$

where $\underline{f} = f_1, \dots, f_n$ are polynomials in $\underline{x} = x_1, \dots, x_n$. Let $\underline{y} = y_1, \dots, y_n$ be another set of variables.
  (a) Show that $\Phi_{\underline{f}}$ is *not surjective* if and only if there is some $\underline{a} \in K^n$ such that the system

$$Z_K(f_1(\underline{x}) - a_1, \dots, f_n(\underline{x}) - a_n) = \varnothing.$$

  has no solution over K.
  (b) Show that $\Phi_{\underline{f}}$ is *injective* if and only if

$$Z_K(f_1(\underline{x}) - f_1(\underline{y}), \dots, f_n(\underline{x}) - f_n(\underline{y})) \subseteq Z_K(x_i - y_i) \qquad \text{for each } i.$$

  (c) Suppose that $K = \mathbb{C}$, and that $\Phi_{\underline{f}}$ is injective but not surjective. Show that there exist $g_i(\underline{x}), h_{i,j}(\underline{x}, \underline{y}) \in \mathbb{C}[\underline{x}, \underline{y}]$ and integers t_j for each j such that

$$\sum_i g_i(\underline{x})(f_i(\underline{x}) - a_i) = 1, \quad (x_j - y_j)^{t_j} = \sum_i h_{i,j}(\underline{x}, \underline{y})(f_i(\underline{x}) - f_i(\underline{y})) \quad \text{in } \mathbb{C}[\underline{x}, \underline{y}].$$

  Setting $R = \mathbb{Z}[\{\text{coefficients of } f_i\text{'s}, g_i\text{'s}, h_{i,j}\text{'s}\}, a_1, \dots, a_n]$, conclude that the same equalities hold in a polynomial ring over R, which in particular is a finitely generated $\mathbb{Z}$-algebra.
  (d) Prove the Ax-Grothendieck Theorem: If $K = \mathbb{C}$, and $\Phi_{\underline{f}}$ as above is injective, then it is surjective.[1]

[1]Hint: Use problem #4!

(Bonus) Let K be a field, $R = K[x_1, \ldots, x_d]$, G be a finite group of order $|G| = N$, and assume that $\mathrm{char}(K) \nmid N$. Let G act linearly on R.

(a) Let $\{f_\sigma \mid \sigma \in G\}$ be N homogeneous elements of R, not necessarily distinct. Consider the element
$$\alpha = \sum_{\tau \in G} \left(\prod_{\sigma \in G} (f_\sigma - \tau\sigma(f_\sigma)) \right).$$
Show that $\alpha = 0$.

(b) Expand out the binomials on in the expression for α, and collect terms to give an expression of the form $\alpha = \sum_{\tau \in G} \prod_{S \subseteq G} \Phi_S$ for some Φ_S. Use this expression to show that $\prod_{\sigma \in G} f_\sigma \in (R^G)_+ \cdot R$. Conclude that $R_{\geq N}$, the ideal of elements of R generated by homogeneous elements of degree at least N, is contained in $(R^G)_+ \cdot R$.

(c) Use part (b) to show that R^G is generated as a K-algebra by elements of degree $\leq N$.

(d) Explain how you can use part (c) to give an algorithm to compute the ring of invariants.

(e) Find a complete set of generators and relations for the ring of invariants of the action in problem #3 of HW #1.