Álgebra Conmutativa, Fall 2019, Homework #1

(1) Let K be a field, and $R := K[x^2, x^3] \subseteq S := K[x]$. Let $I = x^2R$. Show that $IS \cap R \supsetneq I$, and conclude that R is not a direct summand of S.

(2) Let $f_1, \ldots, f_n \in R := \mathbb{C}[x_1, \ldots, x_n]$. Show that if the homomorphism determined by $x_i \mapsto f_i$ is an automorphism of R, then $\det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \in R$ is a nonzero constant.

(3) Let a two generated group $G = \langle \sigma, \tau \rangle$ act on $R := \mathbb{C}[x, y]$ by the rules $\sigma|_{\mathbb{C}} = \tau|_{\mathbb{C}} = \mathrm{id}_{\mathbb{C}}$, and

$$\sigma \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ix \\ -iy \end{pmatrix} \qquad \text{and} \qquad \tau \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

(a) Find two linearly independent invariants of degree 4 in R^G, and a nonzero invariant of degree 6 in R^G.
(b) Find a nonzero relation on the three invariants you found.

(4) Let $R = \dfrac{\mathbb{C}[x, y, u, v]}{(xy - uv)}$. Find three polynomials f_1, f_2, f_3 such that f_1, f_2, f_3 are algebraically independent and $S := \mathbb{C}[f_1, f_2, f_3] \subseteq R$ is module-finite. Find a generating set for R as an S-module.[1]

(5) Let R be a ring, and M an R-module. The *Nagata idealization* of (R, M) is the ring $R \ltimes M$ such that

- as a set, $R \ltimes M = R \times M$;
- the addition is $(r, m) + (s, n) = (r + s, m + n)$;
- the multplication is $(r, m)(s, n) = (rs, sm + rn)$.

(a) Check that $R \ltimes M$ with the operations specified about is a commutative ring.
(b) Let R be a ring that is not Noetherian, and I an ideal that is not finitely generated. Show that[2] $R \subseteq R \ltimes I \subseteq R \ltimes R$, that $R \subseteq R \ltimes R$ is module-finite, but $R \subseteq R \ltimes I$ is not.

(6) This problem is about rings of invariants of infinite groups. Let K be an infinite field, and $R = K[x_1, \ldots, x_n]$. Throughout this problem, G acts linearly on R (so $K \subseteq R^G$).

(a) Let G act on R in such a way that for every $g \in G$ and every monomial $\mu \in R$, $g \cdot \mu$ is a scalar multiple of μ. Show that R^G is generated as a K-vector space, and as a K-algebra, by monomials.
(b) Let $G = (K^\times)^m$ act on R by the rule

$$(\lambda_1, \ldots, \lambda_m) \cdot x_1 = \lambda_1^{a_{11}} \cdots \lambda_m^{a_{m1}} x_1, \quad \ldots \quad , (\lambda_1, \ldots, \lambda_m) \cdot x_n = \lambda_1^{a_{1n}} \cdots \lambda_m^{a_{mn}} x_n,$$

for some $a_{ij} \in \mathbb{Z}$. Let $A = [a_{ij}]$. Show that R^G has as a K-vector space basis the set of monomials $\mu = x_1^{b_1} \cdots x_n^{b_n}$ such that $A \cdot [b_1, \ldots, b_n]^T = 0$ with $b_1, \ldots, b_n \geq 0$.
(c) In the same setting as in part (b), show that R^G is a direct summand of R. Conclude that R^G is a generated as a K-algebra by *finitely many* monomials.
(d) Let $G = K^\times$ act on $K[x, y, z, u, v]$ by

$$\lambda \cdot x = \lambda x, \quad \lambda \cdot y = \lambda y, \quad \lambda \cdot z = \lambda z, \quad \lambda \cdot u = \lambda^{-1} u, \quad \lambda \cdot v = \lambda^{-1} v.$$

Find a set of generators for R^G as a K-algebra.

[1]Hint: Taking three variables won't work, but you can find three linear forms that do work.
[2]Note that R is a subring of $R \ltimes M$ (via the inclusion $r \mapsto (r, 0)$), and as an R-module, $R \ltimes M \cong R \oplus M$.