# What Does it Mean to be True?

Jordan Barrett, Upward Bound Math/Science

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"In principle, we can know all of mathematics. It is given to us in its entirety and does not change. That part of it which we have a perfect view seems beautiful, suggesting harmony; that is that all the parts fit together, though we see fragments of them only. Mathematics is applied to the real world and has proved fruitful. This suggests that the mathematical parts and the empirical parts are in harmony and the real world is also beautiful."

- *Kurt Göedel*

## 1 Statements

In life, you will encounter many different notions of "truth." In Mathematics however, we tend to be very literal.

**Definition 1** *A **statement** is any sentence that has exactly one* truth value*. That is, we may unambiguously label that sentence with **T** or **F**.*

**Example 1** *Consider the following statements*

1. $3^2 + 4^2 = 5^2$
2. *Jordan (your instructor) is six feet tall*
3. *Jordan (your instructor) is less than six feet tall*
4. *There are infinitely many prime numbers of the form* $n^2 + 1$

**Exercise 1** *Can you assign the proper truth values to the above statements? If so, label them with either **T** or **F**.*

hint: *Do not spend too much time thinking about number four*

**Example 2** *The following are examples of non-statements*

1. $|x - 3| \leq 100$
2. *$n$ is an odd number.*
3. *Why are you reading this?*
4. *Steve.*
5. *This sentence is false.*

**Exercise 2** *Why are the sentences above not statements? Can you explain each? Discuss this with Jordan and your classmates.*

## 2 Logical Operations

As you may recall from elementary arithmetic, given two numbers we can combine them in various ways (using "arithmetic operations") to produce a new number.

**Example 3** *I assert that* $2 + 2 = 4$.

Given logical statements, we can combine them to form new statements in a similar way.

**Definition 2** *Let* $P$ *and* $Q$ *be statements. Then by definition, we can label each with either a* ***T*** *or* ***F****. We can organize all possible combinations into a* ***truth table*** *as below:*

| $P$ | $Q$ |
|---|---|
| $T$ | $T$ |
| $T$ | $F$ |
| $F$ | $T$ |
| $F$ | $F$ |

With this idea, we can combine $P$ and $Q$ into a new statement (call it $S$) by specifying the possible truth values of $S$, given the truth values of both "P" and "Q".

**Example 4** *Given statements* $P$ *and* $Q$*, we can define a third statement that we read as "*$P$ *and* $Q$*" as below. We denote "and" by the symbol* $\wedge$

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

**Exercise 3** *Can you similarly define "*$P$ *or* $Q$*"? In math, we denote this by "*$P \vee Q$*". Fill out the following truth table:*

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| $T$ | $T$ | |
| $T$ | $F$ | |
| $F$ | $T$ | |
| $F$ | $F$ | |

Bonus: the next time someone asks "do you want cake or ice cream" what is the best answer?

Another important operation on statements is the idea of "negation." Given a statement $P$ we write its negation as $\neg P$. The truth table is given as below:

| $P$ | $\neg P$ |
|---|---|
| $T$ | $F$ |
| $F$ | $T$ |

**Exercise 4** *Write the negation of the following statements:*

1. *This worksheet was written before 9:00am today.*
2. *Jordan studied math in college and he has brown hair.*
3. *If you are in the United States then you are in Nebraska.*
4. *Jordan's car is blue or green.*

## 3 Conditional Statements

One of the most important types of statements in mathematics are called *conditional statements.* These are statements of the form **if P then Q**, which we denote by $P \Rightarrow Q$. In this form, the statement P is called the *hypothesis* and statement Q is known as the *conclusion.* The truth table for this type of statement is given below:

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

**Example 5** *The following statements are conditional.*

1. *If you are in Nebraska, then you are in the United States.*
2. *If* $x > 0$ *then* $x + 1 > 1$.
3. *If* $n$ *is even, then* $n^2$ *is even.*

**Exercise 5** *Identify the hypotheses and conclusions of the above statements. Can you write your own conditional statement?*

In mathematics, we often seek to *prove* such conditional statements. That is, given a hypothesis which we assume to be true, we seek to develop an argument that demonstrates that our desired conclusion must also be true. That is, we show $P \Rightarrow Q$ is true.

# 4 Proof

**Conjecture 1** *If $n$ is an odd integer, then $n^2 - 1$ is divisible by 8.*

**Exercise 6** *Consider the following questions*

1. *Is this of the form $P \Rightarrow Q$?*
2. *What is $P$? What is $Q$?*
3. *Develop an argument (proof) that shows that Conjecture 1 is true*
4. *Once you have your proof, label Conjecture 1 as "Theorem 1".*
5. *Try to write your own conjecture*