Symmetry and Structure

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"Symmetry is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of the mathematical intellect."

- Hermann Weyl

1 Acting on The Square

Each group has been given a two-sided paper square, with four colored corners on each side. You can imagine this square was cut out of an infinite plane.

Exercise 1 *Establish a fixed starting position for your group's square.*

1. *Observe the "position" of each color.*
2. *Remove the square from its place in the plane.*
3. *Move the square in some way, then place it back in the space it originally occupied.*
4. *Observe the new "position" of each color. Are they the same? Did the positions change?*
5. *If not, devise a move such that the final positions of at least two colors change.*

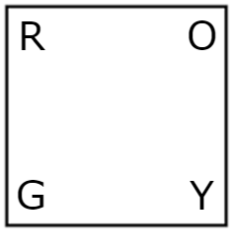
As you likely observed, there are many ways one can accomplish 5. Indeed, it may initially seem as though there are infinitely many options. However, let us consider only the net-motion of the square. That is, we only consider the final position of each color, relative to their starting positions. With this, we find there are finitely many possibilities.

Exercise 2 *Discuss possible moves with your group. How many distinct moves do you believe are possible? Can you develop a logical argument that establishes the exact number?*

hint: The square has two sides, and for each side any given color has four possible positions.

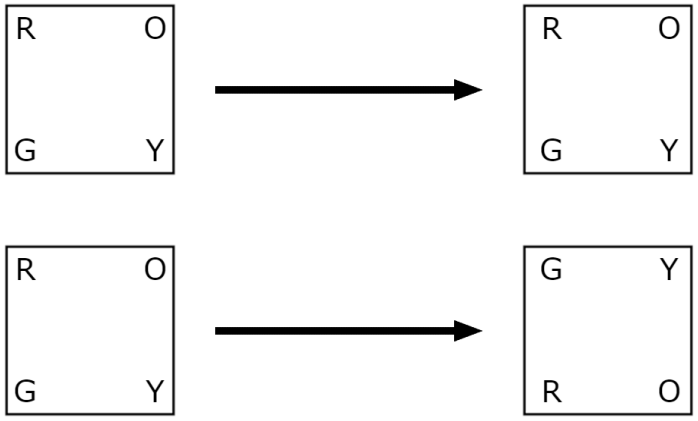
2 Assigning Symbols

In Mathematics, it is often useful to assign names and/or symbols to structures in order to describe them. To study our paper square, we can draw diagrams as below:

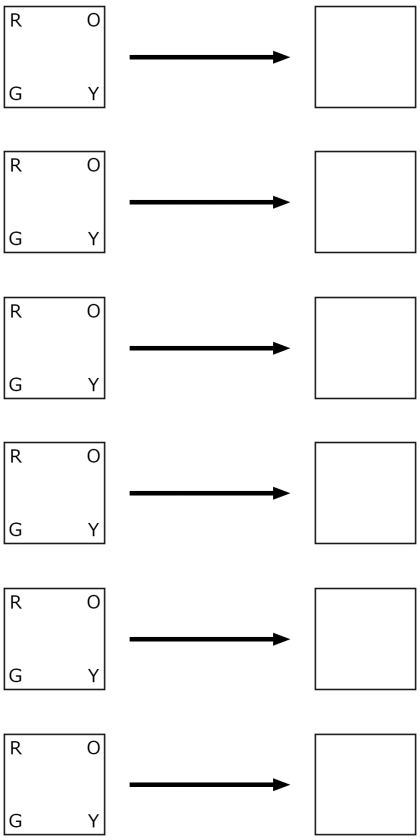


Further, we can use these diagrams to record the net motion of our square.

Example 1 *Below are the diagrams associated with the trivial move, as well as with a horizontal flip.*



Exercise 3 *Record the remaining six possible moves in the diagram below:*



3 Adding Moves

We have now elucidated all eight basic ways of acting on the square. But what happens if you compose these moves? That is, perform them in sequence.

Exercise 4 *Consider the following questions about the composition of moves*

1. *What happens when you compose any move with the trivial move?*
2. *What happens when you compose the horizontal flip with itself?*
3. *How many times do you need to compose the rotation by ninety degrees move to get the trivial move?*
4. *Can you combine flips and rotations to get the trivial move?*

hint: To keep track of these moves, it helps to assign each a name. I recommend R for rotation by ninety degrees, H for the horizontal flip, V for vertical flip, D for flipping along the main diagonal, D' for flipping along the other diagonal and I for the trivial move.

As you likely recall from your elementary education, you can record the results of integer multiplication in a table, called a "multiplication table." We can record our results about the composition of moves in a similar way, using an analogous structure known as a "Cayley Table" (named for English Mathematician Arthur Cayley).

Exercise 5 *Fill the Cayley table below by performing each composition with your paper square, and comparing the result to the eight basic moves.*

	I	R	R^2	R^3	H	V	D	D'
I								
R								
R^2								
R^3								
H								
V								
D								
D'								