# The Tower of Hanoi

## Math Circle

**The Legend**
There is a story about an Indian temple which contains a large room with three time-worn posts in it surrounded by 64 golden disks. Brahmin monks, acting out the command of an ancient prophecy, have been moving these disks, in accordance with the immutable rules of the Brahma, since that time. (The puzzle is sometimes referred to as the Tower of Brahma puzzle.) According to the legend, when the last move of the puzzle will be completed, the world will end.

**The Puzzle**

There are 3 pegs and 7 discs with different diameters. The discs are all stacked on one of the pegs. In the stack, they are ordered by size: smallest on top, largest on the bottom. The goal is to move the entire tower onto one of two free pegs, by the following rules:

- Only one disc may be moved at a time,
- A larger disc may never be stacked onto a smaller disc.

**Question 1.** *Is there a solution to this puzzle? In other words, is there a way to transfer all the discs to another peg according to the given rules?*

**Question 2.** *What is the least number of moves needed to solve this problem?*

**Question 3.** *If the legend is true, when is the world supposed to end?*

**The Mathematics**

Step 1: Experiment. Let's ask a simpler question first. What if we only had 1 disc? 2 discs?

Step 2: Generalize. Instead of 7 discs, what if we ask the question with *n* discs? Generalizing this way allows us to simplify by using what we learned from the small cases first.

| Number of discs | Number of moves |
|---|---|
| **0** | |
| **1** | |
| **2** | |
| **3** | |
| **4** | |
| **5** | |
| ***n*** | |

Step 3: Introduce notation. Let $T_n$ := the minimum number of moves necessary to move $n$ discs according to the rules given. It's fairly easy to see that in our small cases above, we couldn't have solved the puzzle in fewer moves. Therefore, we know that

$T_0 =$ $T_1 =$ $T_2 =$ $T_3 =$

Step 4: Find a pattern. By experimenting again, we can determine that one way to move $n$ discs is to

a.

b.

c.

These three steps require ____, ____, and ____ moves, respectively.

*Can we move the discs with fewer moves?*

No! Because each of the steps listed above are necessary.

We can conclude that:

$$T_0 =$$

$$T_n = \qquad\qquad\qquad \textbf{for } n \geq 1.$$

What we've found at this point is called a "**recurrence relation**." We can use it to compute $T_n$ for any $n$ that we like. For instance, how many moves does it take to transfer 6 discs?

This is a lot of work, so let's see if we can do better.

Step 5: Solve the recurrence. (optional)

To solve the recurrence, we first look at small cases to see if we can discern a pattern:

$T_3 =$ $T_4 =$ $T_5 =$ $T_6 =$

For $n \leq 6$ at least, we see that

$$T_n =$$

To prove that this pattern is true for *all n* (for any number of discs), we have to use a method called "mathematical induction." This is a cool concept you can learn in college math classes :)

One way to think about this method is as a ladder: in order to prove that we can climb a ladder to any height we'd like, we show that we can climb onto the bottom step of the ladder and that once we're on some step of the (infinite) ladder we can always climb up one more step.

So, can we answer the third question now? ***How long will it take to move the 64 discs?***

Assuming that the Brahma monks are able to move the discs at a rate of one per second, using the smallest number of moves, it would take them roughly **585 billion years** or 18,446,744,073,709,551,615 turns to finish, or about 127 times the current age of the sun!!!

The material for this workshop is adapted from §1.1 of *Concrete Mathematics: A Foundation for Computer Science* by Graham, Knuth, Patashnik.

Information about the legend taken from Wikipedia.com.